

# Differentiable Rigidity for quasiperiodic cocycles in compact Lie groups

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## Abstract

We study close-to-constants quasiperiodic cocycles in  $\mathbb{T}^d \times G$ , where  $d \in \mathbb{N}^*$  and  $G$  is a compact Lie group, under the assumption that the rotation in the basis satisfies a Diophantine condition. We prove differentiable rigidity for such cocycles: if such a cocycle is measurably conjugate to a constant one satisfying a Diophantine condition with respect to the rotation, then it is  $C^\infty$ -conjugate to it, and the K.A.M. scheme actually produces a conjugation. We also derive a global differentiable rigidity theorem, assuming the convergence of the renormalization scheme for such dynamical systems.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Facts from algebra and arithmetics</b>	<b>3</b>
2.1	The group $SU(2)$	3
2.2	General compact groups	4
2.3	Calculus and Functional Spaces	5
2.3.1	Functional spaces	5
2.4	Arithmetics, continued fraction expansion	5
<b>3</b>	<b>Cocycles in <math>\mathbb{T}^d \times G</math></b>	<b>6</b>
3.1	The dynamics	6
3.2	Classes of cocycles with simple dynamics, conjugation	7
<b>4</b>	<b>Almost reducibility</b>	<b>8</b>
4.1	Local conjugation in $\mathbb{T}^d \times SU(2)$	8
4.2	Local conjugation in general Lie groups	10
4.3	The K.A.M. scheme	11
4.4	A "K.A.M. normal form"	11

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<b>5</b>	<b>Proof of theorem 1.1</b>	<b>11</b>
5.1	The toy case	12
5.2	Proof of theorem 1.1	12
5.3	Reducibility to a Liouvillean constant	14

## 1 Introduction

In the author's PhD thesis [Kar13], the study of quasiperiodic cocycles in  $\mathbb{T}^d \times G$ , with  $G$  a semisimple compact Lie group, over a Diophantine rotation and satisfying a closeness to constants assumption was revisited. The basic reference for the subject is [Kri99], where the corresponding local density theorem is proved in the  $C^\infty$  category by means of a K.A.M. scheme. The problem of loss of periodicity (i.e. of conjugations of accumulating periods longer than 1 in the presence of resonances) was settled outside the iterative step of the scheme and made necessary the combination of the local almost quasi-reducibility theorem with the reducibility theorem in a positive measure set of parameters in order to obtain a proof of local almost reducibility. We were able to deal with this complication by proving a more efficient local conjugation lemma, in which the phenomenon of longer periods is no longer present, and which can serve as an iterative step of a K.A.M. scheme.

This improved scheme can be used in the proof of a local differentiable rigidity theorem which improves the one obtained in [HY09]. Here,  $G$  is a semisimple compact Lie group.

**Theorem 1.1.** *We suppose that  $\alpha \in DC(\gamma, \tau)$ , and that  $(\alpha, Ae^{F(\cdot)}) \in SW_\alpha^\infty(\mathbb{T}^d, G)$  is close enough to  $(\alpha, A)$  so that the K.A.M. scheme can be initiated. Here,  $A \in G$  is a constant. Moreover, we suppose that there exists  $D(\cdot) : \mathbb{T}^d \rightarrow G$ , measurable, such that  $Conj_{D(\cdot)}(\alpha, Ae^{F(\cdot)}) = (\alpha, A_d)$ , where  $A_d \in DC_\alpha \subset G$ . Then, the K.A.M. scheme can be made (with an appropriate adjustment of a parameter) to produce only a finite number of resonances, and therefore to produce a  $C^\infty$  conjugation. In particular,  $(\alpha, Ae^{F(\cdot)})$  is reducible.*

The Diophantine condition  $DC$  is defined in def. 2.1, and condition  $DC_\alpha$  in 5.1. The proof is carried out in  $\mathbb{T}^d \times SU(2)$ , then extrapolated to general compact Lie group  $G$  by means of an appropriate embedding  $SU(2) \hookrightarrow G$ , so that no background on Lie group theory is demanded for the greatest part of the note. Of course,  $DC$  is of full measure in  $\mathbb{T}^d$ , and for every fixed  $\alpha$  the condition  $DC_\alpha$  is of full Haar measure in  $G$ . The improvement in comparison with the article cited above consists in the more general algebraic context of our theorem, but mainly in the fact that the smallness of the perturbation in our theorem is not related with the measurable conjugation.

Subsequently, we briefly discuss the opposite phenomenon observed for "generic" cocycles smoothly conjugate to Liouville constant cocycles, where the K.A.M. scheme produces an infinite number of resonances, and address the reader to [Kar14] for an improvement of the scheme which settles this problem.

Finally, we use the convergence of renormalization, as well as the measurable invariance of the degree (see [Kar13]) in order to obtain a global differentiable rigidity theorem, without any assumption of closeness to constants, but valid only for one-frequency cocycles ( $d = 1$ ).

**Theorem 1.2.** *We suppose that  $\alpha \in RDC$ , and that  $(\alpha, A(\cdot)) \in SW_\alpha^\infty(\mathbb{T}, G)$  is of degree 0. Moreover, we suppose that there exists  $D(\cdot) : \mathbb{T}^d \rightarrow G$ , measurable, such that  $Conj_{D(\cdot)}(\alpha, A(\cdot)) = (\alpha, A_d)$ , where  $A_d \in DC_\alpha \subset G$ . Then,  $(\alpha, A(\cdot))$  is reducible.*

The assumption that  $\deg(\alpha, A(\cdot)) = 0$ , which we will not define here, assures that renormalization (see [Kri01],[AFK01],[AK06],[Kar13]) converges to constants. This fact, combined with the assumption  $\alpha \in RDC$  (also of full measure in  $\mathbb{T}$ ), implies that there exists  $\tilde{D}(\cdot) \in C^\infty(\mathbb{T}, G)$ , such that  $Conj_{\tilde{D}(\cdot)}(\alpha, A(\cdot))$  satisfies the assumptions of theorem 1.1. The condition  $\deg(\alpha, A(\cdot)) = 0$  is open and dense in the  $C^\infty$ -topology, provided that  $\alpha \in RDC$ . We will not come back to the proof of this theorem, since its details exceed the scope of a short note.

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## 2 Facts from algebra and arithmetics

### 2.1 The group $SU(2)$

The matrix group  $G = SU(2)$  is the multiplicative group of unitary  $2 \times 2$  matrices of determinant 1.

Let us denote the matrix  $S \in G$ ,  $S = \begin{pmatrix} z & w \\ -\bar{z} & \bar{w} \end{pmatrix}$ , where  $(z, w) \in \mathbb{C}^2$  and  $|z|^2 + |w|^2 = 1$ , by  $\{z, w\}_G$ , and the subscript will be omitted unless necessary. The manifold  $G = SU(2)$  is thus naturally identified with  $\mathbb{S}^3 \subset \mathbb{C}^2$ . When coordinates in  $\mathbb{C}^2$  are fixed, the circle  $\mathbb{S}^1$  is naturally embedded in  $G$  as the group of diagonal matrices, which is a maximal torus (i.e. a maximal abelian subgroup) of  $G$ . The center of  $G$ , noted by  $Z_G$  is equal to  $\{\pm Id\}$ .

The Lie algebra  $\mathfrak{g} = \mathfrak{su}(2)$  is naturally isomorphic to  $\mathbb{R}^3 \approx \mathbb{R} \times \mathbb{C}$  equipped with its vector and scalar product. It will be denoted by  $\mathfrak{g}$ . The element  $s = \begin{bmatrix} it & u \\ -\bar{u} & -it \end{bmatrix}$  will be denoted by  $\{t, u\}_g \in \mathbb{R} \times \mathbb{C}$ . Mappings with values in  $\mathfrak{g}$  will be denoted by

$$U(\cdot) = \{U_t(\cdot), U_z(\cdot)\}_g = U_t(\cdot)h + U_z(\cdot)j$$

in these coordinates, where  $U_t(\cdot)$  is a real-valued and  $U_z(\cdot)$  is a complex-valued function. The vectors  $h, j, ij$  form an orthonormal and positively oriented basis for  $\mathfrak{su}(2)$ .

The adjoint action of  $h \in su(2)$  on itself is pushed-forward to twice the vector product:

$$ad_{\{1,0\}}.\{0,1\} = [\{1,0\}, \{0,1\}] = 2\{0,i\} = 2ij$$

plus cyclic permutations, and the Cartan-Killing form, normalized by  $\langle h, h' \rangle = -\frac{1}{8\pi}tr(ad(h) \circ ad(h'))$  is pushed-forward to the scalar product of  $\mathbb{R}^3$ . The preimages of the  $Z_G$  in the maximal toral (i.e. abelian) algebra of diagonal matrices are points of coordinates in the lattice  $\pi\mathbb{Z}$ .

The adjoint action of the group on its algebra is pushed-forward to the action of  $SO(3) \approx SU(2)/\pm Id$  on  $\mathbb{R} \times \mathbb{C}$ . In particular, for diagonal matrices, of the form  $S = \exp(\{2\pi s, 0\}_g)$ ,  $Ad(S).\{t, u\} = \{t, e^{4i\pi s}u\}$ .

## 2.2 General compact groups

The only fact that we will need from the theory of semisimple Lie groups is the decomposition of the Lie algebra,  $\mathfrak{g}'$ , of such a group  $G'$  in factors isomorphic to  $\mathfrak{g} = su(2)$ . If  $\mathfrak{g}'$  is such an algebra, then for any maximal abelian algebra  $\mathfrak{t}$  (i.e. a subalgebra on which the restriction of the Lie bracket vanishes identically), then the orthogonal complement of  $\mathfrak{t}$  admits a decomposition into a direct sum of mutually orthogonal 2-dimensional real subspaces  $E_\rho$ ,  $\rho \in \Delta_+$ ,<sup>1</sup> for which the following holds. For each  $E_\rho$ , there exists  $h_\rho \in \mathfrak{t}$  such that  $\mathbb{R}h_\rho \oplus E_\rho \approx su(2)$ . One can choose a vector  $j_\rho \in E_\rho$ , such that the vectors  $h_\rho, j_\rho, ij_\rho$  form a basis of  $\mathbb{R}h_\rho \oplus E_\rho$ , satisfying the same relations as the basis of  $su(2)$ .

There exists, moreover,  $\tilde{\Delta} \subset \Delta_+$ , such that  $(h_\rho)_{\tilde{\Delta}}$  is a basis for  $\mathfrak{t}$ , for which there exists  $P \in \mathbb{N}^*$  and  $m_{\rho, \rho'} \in \mathbb{N}$  such that, for all  $\rho' \in \Delta_+$ ,

$$h_{\rho'} = \frac{1}{P} \sum_{\rho \in \Delta} m_{\rho, \rho'} h_{\rho} \quad (1)$$

The set of roots  $\Delta_+$  satisfies the property that for all  $\rho \in \Delta_+$ , and for all  $a \in \mathfrak{t}$ ,  $[a, j_\rho] = 2i\pi\rho(a)j_\rho = 2i\pi a_\rho j_\rho$ . The adjoint action of  $G$  on  $\mathfrak{t}$  (and therefore on the vectors  $h_\rho$ ) induces an action of  $G$  on  $\Delta_+$ .

The Lie algebra  $\mathfrak{g}$  is thus decomposed into

$$\mathfrak{c} \oplus \mathfrak{t} \oplus \left( \bigoplus_{\rho \in \Delta_+} E_\rho \right) = \mathfrak{c} \oplus \bigoplus_{\rho \in \tilde{\Delta}} \mathbb{R}h_\rho \oplus \bigoplus_{\rho \in \Delta_+} \mathbb{C}j_\rho$$

Here,  $\mathfrak{c}$  is the center of the algebra, which is trivial if the algebra is semisimple. This decomposition is referred to as "root space decomposition with respect to the toral algebra  $\mathfrak{t}$ ".

The lattice of preimages of  $Z_G$  in  $\mathfrak{t}$  will be denoted by  $\mathcal{Z}$ .

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<sup>1</sup>The finite set  $\Delta_+$  is called the set of positive roots of  $\mathfrak{g}'$ , and it is a subset of  $\mathfrak{t}^*$

## 2.3 Calculus and Functional Spaces

### 2.3.1 Functional spaces

We will consider the space  $C^\infty(\mathbb{T}^d, g)$  equipped with the standard maximum norms

$$\|U\|_s = \max_{0 \leq \sigma \leq s} \max_{\mathbb{T}^d} |\partial^\sigma U(\cdot)|$$

for  $s \geq 0$ , and the Sobolev norms

$$\|U\|_{H^s}^2 = \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^s |\hat{U}(k)|^2$$

where  $\hat{U}(k) = \int U(\cdot) e^{-2i\pi kx}$  are the Fourier coefficients of  $U(\cdot)$ . The fact that the injections  $H^{s+d/2}(\mathbb{T}, g) \hookrightarrow C^s(\mathbb{T}^d, g)$  and  $C^s(\mathbb{T}^d, g) \hookrightarrow H^s(\mathbb{T}^d, g)$  for all  $s \geq 0$  are continuous is classical.

We will also use the convexity or Hadamard-Kolmogorov inequalities (see [Kol49]) ( $U \in C^\infty(\mathbb{T}, g)$ ):

$$\|U(\cdot)\|_\sigma \leq C_{s,\sigma} \|U\|_0^{1-\sigma/s} \|U\|_s^{\sigma/s}$$

for  $0 \leq \sigma \leq s$ , and the inequalities concerning the composition of functions (see [Kri99]):

$$\|\phi \circ (f + u) - \phi \circ f\|_s \leq C_s \|\phi\|_{s+1} (1 + \|f\|_0)^s (1 + \|f\|_s) \|u\|_s$$

We will use the truncation operators for mappings  $\mathbb{T}^d \rightarrow g$  defined by

$$\begin{aligned} T_N f(\cdot) &= \sum_{|k| \leq N} \hat{f}(k) e^{2i\pi k \cdot} \\ \dot{T}_N f(\cdot) &= T_N f(\cdot) - \hat{f}(0) \\ R_N f(\cdot) &= \sum_{|k| > N} \hat{f}(k) e^{2i\pi k \cdot} \end{aligned}$$

These operators satisfy the estimates

$$\|T_N f(\cdot)\|_{C^s} \leq C_s N^{d/2} \|f(\cdot)\|_{C^s} \quad (2)$$

$$\|R_N f(\cdot)\|_{C^s} \leq C_{s,s'} N^{s-s'+d/2} \|f(\cdot)\|_{C^{s'}} \quad (3)$$

The Fourier spectrum of a function will be denoted by  $\hat{\sigma}(f) = \{k \in \mathbb{Z}^d, \hat{f}(k) \neq 0\}$ .

## 2.4 Arithmetics, continued fraction expansion

The following notion is essential in K.A.M. theory. It is related with the quantification of the closeness of rational numbers to certain classes of irrational numbers.

**Definition 2.1.** We will denote by  $DC(\gamma, \tau)$  the set of numbers  $\alpha$  in  $\mathbb{T} \setminus \mathbb{Q}$  such that for any  $k \neq 0$ ,  $|\alpha k|_{\mathbb{Z}} \geq \frac{\gamma^{-1}}{|k|^{\tau}}$ . Such numbers are called *Diophantine*.

The set  $DC(\gamma, \tau)$ , for  $\tau > 2$  fixed and  $\gamma \in \mathbb{R}_+^*$  is of positive Haar measure in  $\mathbb{T}$ . If we fix  $\tau$  and let  $\gamma$  run through the positive real numbers, we obtain  $\cup_{\gamma>0} DC(\gamma, \tau)$  which is of full Haar measure. The numbers that do not satisfy any Diophantine condition are called *Liouvillean*. They form a residual set of 0 Lebesgue measure.

This last following definition concerns the relation of the approximation of an irrational number with its continued fractions representation.

**Definition 2.2.** We will denote by  $RDC(\gamma, \tau)$  is the set of recurrent Diophantine numbers, i.e. the  $\alpha$  in  $\mathbb{T} \setminus \mathbb{Q}$  such that  $G^n(\alpha) \in DC(\gamma, \tau)$  for infinitely many  $n$ .

Here,  $G(\alpha) = \{\alpha^{-1}\}$  is the Gauss map ( $\{\cdot\}$  stands for "fractional part"). The set  $RDC$  is also of full measure, since the Gauss map is ergodic with respect to a smooth measure.

In contexts where the parameters  $\gamma$  and  $\tau$  are not significant, they will be omitted in the notation of both sets.

### 3 Cocycles in $\mathbb{T}^d \times G$

#### 3.1 The dynamics

Let  $\alpha \in \mathbb{T}^d \equiv \mathbb{R}^d / \mathbb{Z}^d$ ,  $d \in \mathbb{N}^*$ , be an irrational rotation, so that the translation  $x \mapsto x + \alpha \bmod (\mathbb{Z}^d)$  is minimal and uniquely ergodic. The translation will sometimes be denoted by  $R_\alpha$ .

If we also let  $A(\cdot) \in C^\infty(\mathbb{T}^d, G)$ , the couple  $(\alpha, A(\cdot))$  acts on the fibered space  $\mathbb{T}^d \times G \rightarrow \mathbb{T}^d$  defining a diffeomorphism by

$$(\alpha, A(\cdot)).(x, S) = (x + \alpha, A(x).S), (x, S) \in \mathbb{T}^d \times G$$

We will call such an action a *quasiperiodic cocycle over  $R_\alpha$*  (henceforth simply a cocycle). The space of such actions is denoted by  $SW_\alpha^\infty(\mathbb{T}^d, G) \subset Diff^\infty(\mathbb{T}^d \times G)$ . Most times we will abbreviate the notation to  $SW_\alpha^\infty$ . Cocycles are a class of fibered diffeomorphisms, since fibers of  $\mathbb{T}^d \times G$  are mapped into fibers, and the mapping from one fiber to another in general depends on the base point. The number  $d \in \mathbb{N}^*$  is the number of frequencies of the cocycle.

If we consider a representation of  $G$  on a vector space  $E$ , the action of the cocycle can be also defined on  $\mathbb{T}^d \times E$ , simply by replacing  $S$  by a vector in  $E$  and multiplication in  $G$  by the action. The particular case which will be important in this article is the representation of  $G$  on  $g$ , and the resulting action of the cocycle on  $\mathbb{T}^d \times g$ .

The  $n$ -th iterate of the action is given by

$$(\alpha, A(\cdot))^n.(x, S) = (n\alpha, A_n(\cdot)).(x, S) = (x + n\alpha, A_n(x).S)$$

where  $A_n(\cdot)$  represents the *quasiperiodic product* of matrices equal to

$$A_n(\cdot) = \begin{cases} A(\cdot + (n-1)\alpha) \cdots A(\cdot) & , n > 0 \\ Id & , n = 0 \\ A^*(\cdot + n\alpha) \cdots A^*(\cdot - \alpha) & , n < 0 \end{cases}$$

### 3.2 Classes of cocycles with simple dynamics, conjugation

The cocycle  $(\alpha, A(\cdot))$  is called a constant cocycle if  $A(\cdot) = A \in G$  is a constant mapping. In that case, the quasiperiodic product reduces to a simple product of matrices,  $(\alpha, A)^n = (n\alpha, A^n)$ .

The group  $C^\infty(\mathbb{T}^d, G) \hookrightarrow SW^\infty(\mathbb{T}^d, G)$  acts by *dynamical conjugation*: Let  $B(\cdot) \in C^\infty(\mathbb{T}^d, G)$  and  $(\alpha, A(\cdot)) \in SW^\infty(\mathbb{T}^d, G)$ . Then we define

$$\begin{aligned} Conj_{B(\cdot)}(\alpha, A(\cdot)) &= (0, B(\cdot)) \circ (\alpha, A(\cdot)) \circ (0, B(\cdot))^{-1} \\ &= (\alpha, B(\cdot + \alpha).A(\cdot).B^{-1}(\cdot)) \end{aligned}$$

which is in fact a change of variables within each fiber of the product  $\mathbb{T}^d \times G$ . The dynamics of  $Conj_{B(\cdot)}(\alpha, A(\cdot))$  and  $(\alpha, A(\cdot))$  are essentially the same, since

$$(Conj_{B(\cdot)}(\alpha, A(\cdot)))^n = (n\alpha, B(\cdot + n\alpha).A_n(\cdot).B^{-1}(\cdot))$$

**Definition 3.1.** *Two cocycles  $(\alpha, A(\cdot))$  and  $(\alpha, \tilde{A}(\cdot))$  in  $SW_\alpha^\infty$  are conjugate iff there exists  $B(\cdot) \in C^s(\mathbb{T}^d, G)$  such that  $(\alpha, \tilde{A}(\cdot)) = Conj_{B(\cdot)}(\alpha, A(\cdot))$ . We will use the notation  $(\alpha, A(\cdot)) \sim (\alpha, \tilde{A}(\cdot))$  to state that the two cocycles are conjugate to each other.*

Since constant cocycles are a class for which dynamics can be analysed, we give the following definition.

**Definition 3.2.** *A cocycle will be called reducible iff it is conjugate to a constant.*

Due to the fact that not all cocycles are reducible (e.g. generic cocycles in  $\mathbb{T} \times \mathbb{S}^1$  over Liouvillean rotations, but also cocycles over Diophantine rotations, even though this result is hard to obtain, see [Eli02], [Kri01]) we also need the following concept, which has proved to be crucial in the study of such dynamical systems.

**Definition 3.3.** *A cocycle  $(\alpha, A(\cdot))$  is said to be almost reducible if there exists a sequence of conjugations  $B_n(\cdot) \in C^\infty$ , such that  $Conj_{B_n(\cdot)}(\alpha, A(\cdot))$  becomes arbitrarily close to constants in the  $C^\infty$  topology, i.e. iff there exists  $(A_n)$ , a sequence in  $G$ , such that*

$$A_n^*(B_n(\cdot + \alpha)A(\cdot)B_n^*(\cdot)) = e^{F_n(\cdot)} \xrightarrow{C^\infty} Id$$

This property will herein be established in a K.A.M. constructive way, making it possible to measure the rate of convergence versus the explosion of the conjugations. Almost reducibility then comes along with obtaining that

$$Ad(B_n(\cdot)).F_n(\cdot) = B_n(\cdot).F_n(\cdot).B_n^*(\cdot) \xrightarrow{C^\infty} 0$$

If this additional condition is satisfied, almost reducibility in the sense of the definition above and almost reducibility in the sense that "the cocycle can be conjugated arbitrarily close to reducible cocycles" are equivalent.

We can now recall the local almost reducibility theorem. It is used the proof of the local density theorem, already proved in [Kri99]. We will follow the proof in [Kar13], since it implies a useful corollary (cor. 4.2) in a more direct manner than the previously existing proofs.

**Theorem 3.1.** *Let  $\alpha \in DC(\gamma, \tau) \subset \mathbb{T}^d$ ,  $d \geq 1$  and  $G$  a semisimple compact Lie group. Then, there exists  $s_0 \in \mathbb{N}^*$  and  $\epsilon > 0$ , such that if  $(\alpha, Ae^{F(\cdot)}) \in SW_\alpha^\infty(\mathbb{T}^d, )$  with  $\|F(\cdot)\|_0 < \epsilon$  and  $\|F(\cdot)\|_{s_0} < 1$ ,  $(\alpha, Ae^{F(\cdot)})$  is almost reducible.*

## 4 Almost reducibility

In this section we present the basic points of the proof of the thm 3.1. For the next paragraph,  $G = SU(2)$ , and the proof of the local conjugation lemma when  $G$  is an arbitrary compact Lie group will be hinted in the next one.

### 4.1 Local conjugation in $\mathbb{T}^d \times SU(2)$

Let  $(\alpha, Ae^{F(\cdot)}) = (\alpha, A_1 e^{F_1(\cdot)}) \in SW^\infty(\mathbb{T}, G)$  be a cocycle over a Diophantine rotation satisfying some smallness conditions to be made more precise later on. Without any loss of generality, we can also suppose that  $A = \{e^{2i\pi a}, 0\}$  is diagonal. The goal is to conjugate the cocycle ever closer to constant cocycles by means of an iterative scheme. This is obtained by iterating the following lemma, for the detailed proof of which we refer to [Kri99], [Eli02] or [Kar13]. The following lemma is the cornerstone of the procedure, since it represents one step of the scheme. The rest of this paragraph is devoted to a summary of its proof, for the sake of completeness.

**Lemma 4.1.** *Let  $\alpha \in DC(\gamma, \tau)$  and  $K \geq C\gamma N^\tau$ . Let, also,  $(\alpha, Ae^{F(\cdot)}) \in SW^\infty(\mathbb{T}_d, G)$  with*

$$c_{1,0}KN^{s_0}\varepsilon_0 < 1$$

*for some  $s_0 \in \mathbb{N}^*$  depending on  $d, \gamma, \tau$ , and where  $\varepsilon_s = \|F\|_s$ . Then, there exists a conjugation  $G(\cdot) \in C^\infty(\mathbb{T}, G)$  such that*

$$G(\cdot + \alpha).A.e^{F(\cdot)}.G^*(\cdot) = A'e^{F'(\cdot)} \quad (4)$$

*The mappings  $G(\cdot)$  and  $F'(\cdot)$  satisfy the following estimates*

$$\begin{aligned} \|G(\cdot)\|_s &\leq c_{1,s}(N^s + KN^{s+\tau+1/2}\varepsilon_0) \\ \varepsilon'_s &\leq c_{2,s}K^2N^{2\tau+d}(N^s\varepsilon_0 + \varepsilon_s)\varepsilon_0 + C_{s,s'}N^{s-s'+2\tau+d}\varepsilon_{s'} \end{aligned}$$

If we suppose that  $Y(\cdot) : \mathbb{T} \rightarrow g$  can conjugate  $(\alpha, Ae^{F(\cdot)})$  to  $(\alpha, A'e^{F'(\cdot)})$ , with  $\|F'(\cdot)\| \ll \|F(\cdot)\|$ , then it must satisfy the functional equation

$$A^*e^{Y(\cdot+\alpha)}Ae^{F(\cdot)}e^{-Y(\cdot)} = A^*A'e^{F'(\cdot)}$$



Linearization of this equation under the assumption that all  $C^0$  norms are smaller than 1 gives

$$Ad(A^*)Y(\cdot + \alpha) + F(\cdot) - Y(\cdot) = \exp^{-1}(A^*A')$$

which we will write in coordinates, separating the diagonal from the non-diagonal part.

The equation for the diagonal coordinate reads  $Y_t(\cdot + \alpha) - Y_t(\cdot) = -F_t(\cdot)$ . For reasons well known in K.A.M. theory, we have to truncate at an order  $N$  to be determined by the parameters of the problem and obtain a solution to the equation

$$Y_t(\cdot + \alpha) - Y_t(\cdot) = -\dot{T}_N F_t(\cdot)$$

satisfying the estimate  $\|Y_t(\cdot)\|_s \leq \gamma C_s N^{s+\tau+d/2} \|F_t(\cdot)\|_0$ . The rest satisfies the estimate of eq. 3. The mean value  $\hat{F}_t(0)$  is an *obstruction* and will be integrated in  $\exp^{-1}(A^*A')$ .

As for the equation concerning the non-diagonal part, it reads

$$e^{-4i\pi a} Y_z(\cdot + \alpha) - Y_z(\cdot) = -F_z(\cdot) \quad (5)$$

or, in the frequency domain,

$$(e^{2i\pi(k\alpha-2a)} - 1) \hat{Y}_z(k) = -\hat{F}_z(k), \quad k \in \mathbb{Z}^d \quad (6)$$

Therefore, the Fourier coefficient  $\hat{F}_z(k_r)$  cannot be eliminated with good estimates if

$$|k_r \alpha - 2a|_{\mathbb{Z}} < K^{-1}$$

for some  $K > 0$  big enough. If  $K = N^\nu$ , with  $\nu > \tau$ , then we know by [Eli02] that, if such a  $k_r$  exists (called a *resonant mode*) and satisfies  $0 \leq |k_r| \leq N$ , it is unique in  $\{k \in \mathbb{Z}^d, |k - k_r| \leq 2N\}$ . Therefore, if we call  $T_{2N}^{k_r}$  the truncation operator projecting on the frequencies  $0 < |k - k_r| \leq 2N$  if  $k_r$  exists (or on  $|k| \leq N$  if it does not, but this easier and follows from this one), the equation

$$e^{-4i\pi a} Y_z(\cdot + \alpha) - Y_z(\cdot) = -T_{2N}^{k_r} F_z(\cdot)$$

can be solved and the solution satisfies  $\|Y_z(\cdot)\|_s \leq C_s N^{s+\nu+d/2} \|F_z(\cdot)\|_0$ . We will define the rest operator by projection on modes satisfying  $|k - k_r| > 2N$ .

In total, the equation that can be solved with good estimates is

$$Ad(A^*)Y(\cdot + \alpha) - Y(\cdot) = -F(\cdot) + \{\hat{F}_t(0), \hat{F}_z(k_r)e^{2i\pi k_r \cdot}\} + \{R_N F_t(\cdot), R_{2N}^{k_r} F_z(\cdot)\}$$

with  $\|Y(\cdot)\|_s \leq C_s N^{s+\nu+1/2} \|F(\cdot)\|_0$ . Under the smallness assumptions of the hypothesis, the linearization error is small and the conjugation thus constructed satisfies

$$e^{Y(\cdot+\alpha)} A e^{F(\cdot)} e^{-Y(\cdot)} = \{e^{2i\pi(a+\hat{F}_t(0))}, 0\}_G e^{\{0, \hat{F}_z(k_r)e^{2i\pi k_r \cdot}\}_g} e^{\tilde{F}(\cdot)}$$

with  $\tilde{F}(\cdot)$  a "quadratic" term. We remark that, a priori, the obstruction  $\{0, \hat{F}_z(k_r)e^{2i\pi k_r \cdot}\}$  is of the order of the initial perturbation and therefore what we called  $\exp^{-1}(A^*A')$  is not constant in the presence of resonant modes.

If  $k_r$  exists and is non-zero, then application the lemma cannot be iterated. On the other hand, the conjugation  $B(\cdot) = \{e^{-2i\pi k_r \cdot / 2}, 0\}$  is such that, if we call  $F''(\cdot) = Ad(B(\cdot)).F(\cdot) = \{F_t(\cdot), e^{-2i\pi k_r \cdot} F_z(\cdot)\}$ , and  $Y'(\cdot) = Ad(B(\cdot)).Y(\cdot)$ , and  $A'' = B(\alpha)A = \{e^{2i\pi(a-k_r\alpha/2)}, 0\}$ , they satisfy the equation

$$\begin{aligned} Ad((A'')^*)Y'(\cdot + \alpha) - Y'(\cdot) + F_1''(\cdot) &= \{\hat{F}_t(0), \hat{F}_z(k_r)\} + \{R_{2N}F_t(\cdot), e^{-2i\pi k_r \cdot} R_{2N}^{k_r} F_z(\cdot)\} \\ &= \{\hat{F}_t''(0), \hat{F}_z''(0)\} + \{R_{2N}F_t''(\cdot), \tilde{R}_{2N}^{k_r} F_z(\cdot)\} \end{aligned}$$

where  $\tilde{R}_{2N}^{k_r}$  is a dis-centred rest operator, whose spectral support is outside  $[-N, N]^d \cap \mathbb{Z}^d$ , and can therefore be estimated like a classical rest operator  $R_N$ . The equation for primed variables can be obtained from eq. 5 by applying  $Ad(B(\cdot))$  and using that  $B(\cdot)$  is a morphism and commutes with  $A$ . The passage from one equation to the other is equivalent to the fact that

$$\begin{aligned} Conj_{B(\cdot)}(\alpha, A. \exp(\{\hat{F}_t(0), \hat{F}_z(k_r)e^{2i\pi k_r \cdot}\})) &= (\alpha, B(\alpha)A. \exp(\{\hat{F}_t(0), \hat{F}_z(k_r)\})) \\ &= (\alpha, \tilde{A}) \end{aligned}$$

that is,  $B(\cdot)$  reduces the initial constant perturbed by the obstructions to a cocycle close to  $(\alpha, \pm Id)$ . There is a slight complication, as  $B(\cdot)$  may be 2-periodic (if  $\{i\pi k_r, 0\} \in g$  is a preimage of  $-Id$ ). If it is so, we can conjugate a second time with a minimal geodesic  $C(\cdot) : 2\mathbb{T} \rightarrow G$  such that  $C(1) = -Id \in Z_G$  and commuting with  $\tilde{A}$ . The cocycle that we obtain in this way is 1-periodic and close to  $(\alpha, \{e^{i\pi\alpha}, 0\}_G)$ , and the conjugation is also 1-periodic.

Summing up, if we call  $G(\cdot) = C(\cdot)B(\cdot)e^{Y(\cdot)}$  and  $A' = C(\alpha)\tilde{A}$ , there exists  $F'(\cdot)$  satisfying the estimates of the lemma and such that eq. 4 is verified.

## 4.2 Local conjugation in general Lie groups

Local conjugation in a compact group  $G$  is in fact a vector-space version of the lemma of the previous paragraph. The statement is the same, just replacing  $SU(2)$  by a compact group  $G$ , and the steps of the proof are the same. Firstly, solution of the linear equation, which is done in  $\mathfrak{c} \oplus \mathfrak{t}$  as in the diagonal coordinates here above, and in each  $E_\rho$  as in eq. 6. This procedure produces a "vector"  $^2 (k_\rho)_{\rho \in \Delta_+} \in \bigsqcup_{\Delta_+} \mathbb{Z}$  of resonant modes, which are reduced in the second step, where we construct a vector  $H$  such that

$$\begin{aligned} Conj_{\exp(H \cdot)}(\alpha, A. \exp(\{ObF(\cdot)\})) &= (\alpha, e^{H\alpha} A. \exp(\{\hat{F}_{\mathfrak{t} \oplus \mathfrak{c}}(0) + Ad(e^{H \cdot}).ObF(\cdot)\})) \\ &= (\alpha, \tilde{A}) \end{aligned}$$

Here,  $Ob$  stands for projection on the resonant modes,  $\sum_{\rho \in \Delta_+} \hat{F}(k_\rho)e^{2i\pi \cdot} j_\rho$ . The vector  $J \in \frac{1}{P}\mathcal{Z}$  is constructed by solving the linear system of eq. 1 for the vector  $(k_\rho)_{\rho \in \Delta_+}$ . Finally, since the mapping  $\exp(J \cdot)$  may be  $\frac{1}{P\#Z_G}$ -periodic, we post-conjugate with  $C(\cdot)$ , a minimal geodesic connecting the  $Id$  with  $\exp J$  (which measures the failure of  $\exp(J \cdot)$  to be 1-periodic) and commuting with  $\tilde{A}$ . We thus obtain  $A'$  which is close to an element of  $\alpha \frac{1}{P}\mathcal{Z}$ .<sup>3</sup>

<sup>2</sup>Some entries may be  $\emptyset$ , if there is no resonant mode in the corresponding eigenspace, or the corresponding frequency in  $\mathbb{Z}$

<sup>3</sup>Intersected with the ball in  $g$  where  $\exp$  is bijective.

### 4.3 The K.A.M. scheme

Lemma 4.1 can serve as the step of a K.A.M. scheme, with the following standard choice of parameters:  $N_{n+1} = N_n^{1+\sigma} = N^{(1+\sigma)^{n-1}}$ , where  $N = N_1$  is big enough and  $0 < \sigma < 1$ , and  $K_n = N_n^\nu$ , for some  $\nu > \tau$ . If we suppose that  $(\alpha, A_n e^{F_n(\cdot)})$  satisfies the hypotheses of lemma 4.1 for the corresponding parameters, then we obtain a mapping  $G_n(\cdot) = C_n(\cdot).B_n(\cdot)e^{Y_n(\cdot)}$  that conjugates it to  $(\alpha, A_{n+1} e^{F_{n+1}(\cdot)})$ , and we use the notation  $\varepsilon_{n,s} = \|F_n\|_s$ .

If we suppose that the initial perturbation is small in small norm:  $\varepsilon_{1,0} < \epsilon < 1$ , and not big in some bigger norm:  $\varepsilon_{1,s_0} < 1$ , where  $\epsilon$  and  $s_0$  depend on the choice of parameters, then we can prove (see [Kar13] and, through that, [FK09]), that the scheme can be iterated, and moreover

$$\begin{aligned}\varepsilon_{n,s} &= O(N_n^{-\infty}) \text{ for every fixed } s \text{ and} \\ \|G_n\|_s &= O(N_n^{s+\lambda}) \text{ for every } s \text{ and some fixed } \lambda > 0\end{aligned}$$

We say that the norms of perturbations decay exponentially, while conjugations grow polynomially.

### 4.4 A "K.A.M. normal form"

The product of conjugations  $H_n = G_n \cdots G_1$ , which by construction satisfies  $\text{Conj}_{H_n}(\alpha, A_1 e^{F_1}) = (\alpha, A_{n+1} e^{F_{n+1}})$ , is not expected to converge. In fact, it converges iff  $B_n(\cdot) \equiv \text{Id}$ , except for a finite number of steps. Anyhow, we can obtain a "K.A.M. normal form" for cocycles close to constants

**Lemma 4.2.** *Let the hypotheses of theorem 3.1 hold. Then, there exists  $D(\cdot) \in C^\infty(\mathbb{T}, G)$  such that, if we call  $\text{Conj}_{D(\cdot)}(\alpha, A e^{F(\cdot)}) = (\alpha, A' e^{F'(\cdot)})$ , then the K.A.M. scheme applied to  $(\alpha, A' e^{F'(\cdot)})$  for the same choice of parameters consists only in the reduction of resonant modes: The resulting conjugation  $H_{n_i}(\cdot)$  has the form  $\prod_{n_i}^1 C_{n_i}(\cdot).B_{n_i}(\cdot)$ , where  $\{n_i\}$  are the steps in which reduction of a resonant mode took place.*

The proof, for which we refer the reader to [Kar14], is short and uses the fast convergence of the scheme. For a cocycle in normal form, we relabel the indexes as  $(\alpha, A_{n_i} e^{F_{n_i}}) = (\alpha, A_i e^{F_i})$ .

## 5 Proof of theorem 1.1

In this section, we use the K.A.M. scheme outlined above. We begin by a brief study of the rigidity of conjugation between constant cocycles in general compact groups, which is to be compared with section 2.5.e of [Kri99], and shows that conjugation between constant cocycles is rigid, with no assumptions on the arithmetics.

## 5.1 The toy case

Let  $B : \mathbb{T} \rightarrow G$  be a measurable mapping,  $\alpha \in \mathbb{T}^d$  a minimal translation, and  $C_1, C_2 \in G$  such that

$$B(\cdot + \alpha)C_1B^*(\cdot) = C_2$$

By composing  $B$  with a constant if necessary, we can suppose that  $C_1$  and  $C_2$  are on the same maximal torus. If, for simplicity in notation we identify  $G$  and  $\text{Inn}(g) \approx G/Z_G$ , the group which acts by the adjoint action on  $g$ , even though the identification is not accurate, we find that

$$e^{2i\pi k\alpha} \hat{B}(k)C_1 = C_2 \hat{B}(k)$$

so that, if  $e^{2i\pi c_\rho^{(i)}}$  are the eigenvalues of the adjoint action of  $C_i$ , the equation

$$\langle e^{2i\pi(k\alpha + c_\rho^{(1)})} \hat{B}(k)j_\rho, j_{\rho'} \rangle = \langle \hat{B}(k)j_\rho, e^{2i\pi c_\rho^{(2)}} j_{\rho'} \rangle$$

has at most one non-zero solution in  $k$  for each pair of roots  $\rho$  and  $\rho'$  for which there exists  $Z \in G$  such that  $\rho' = Z.\rho$ . Using a similar argument for directions in the torus, we find that  $B : \mathbb{T} \rightarrow \text{Inn}(g)$ , and therefore  $B : \mathbb{T} \rightarrow G$ , has a finite support in the frequency space, and therefore it is  $C^\infty$ . Finally, it can be easily proved that  $B(\cdot)$  can be written as a product of two commuting morphisms, which do not take values in the same maximal torus.

## 5.2 Proof of theorem 1.1

In order to simplify the proof and to avoid the phenomena related to loss of periodicity, we consider a unitary representation of  $G$  and we suppose firstly that  $D(\cdot + \alpha)A_1e^{F_1(\cdot)}D^*(\cdot) = A_d$  with  $D(\cdot) : \mathbb{T} \rightarrow G \hookrightarrow U(w)$  a measurable mapping and  $F_1(\cdot)$  small enough so that the reduction scheme can be applied. The K.A.M. scheme of the previous section, applied in the simpler algebraic context of  $U(w)$ , produces the sequence of conjugations  $H_{n_i}(\cdot) = H_i(\cdot) = B_{n_i}(\cdot) \cdots B_{n_1}(\cdot)$ . This product converges if, and only if, it is finite, and satisfies

$$D(\cdot + \alpha)H_i^*(\cdot + \alpha)A_i e^{F_i(\cdot)} H_i(\cdot) D^*(\cdot) = A_d$$

or, by introducing some obvious notation

$$D_i(\cdot + \alpha)A_i e^{F_i(\cdot)} D_i^*(\cdot) = A_d$$

Finally, by post-conjugating with a constant we can assume that  $A_i$  and  $A_d$  are in the same maximal torus.

**The case  $G = SU(2)$**

Let us firstly examine the simpler case where  $G = SU(2)$ , and suppose that  $H_i(\cdot)$  diverges. If we call  $D_i^{j,j}(\cdot) = \langle D_i(\cdot)j, j \rangle : \mathbb{T}^d \rightarrow \mathbb{C}$ , we have

$$(e^{2i\pi(k\alpha + a_i - a_d)} - 1)e^{2i\pi a_d} \hat{D}_i^{j,j}(k) = O(\varepsilon_{n,0})$$

where  $O(\varepsilon_{n,0})$  is bounded independently of  $k$ . The divergence of  $D_i(\cdot)$  in  $L^2$  implies that  $|a_i|_{\mathbb{Z}} \leq K_{n_i}^{-1} = N_{n_i}^{-\nu}$ , where  $A_i = \{\exp(2i\pi a_i), 0\}_{SU(2)}$ . Therefore,

$$|k\alpha + a_i - a_d|_{\mathbb{Z}} \geq K_{n_i}, \forall 0 < |k| \leq (2\tilde{\gamma}^{-1})^{1/\tau} N_{n_i}^{\nu/\tau} = N'_i$$

provided that

$$|k\alpha - a_d|_{\mathbb{Z}} \geq \frac{\tilde{\gamma}^{-1}}{|k|^\tau} \quad (7)$$

for  $k \in \mathbb{Z}^*$ , i.e. iff  $a_d \in DC_\alpha(\tilde{\gamma}, \tau)$ . This motivates the following definition.

**Definition 5.1.** A constant  $A = \exp(a) \in G$  is diophantine with respect to  $\alpha$  iff for all roots  $\rho$  of a root-space decomposition with respect to a torus passing by  $A$  we have  $\rho(a) \in DC_\alpha(\tilde{\gamma}, \tilde{\tau})$ , i.e. if eq. 7 is satisfied the constants  $\gamma, \tilde{\tau}$ . By abuse of notation we will write  $A \in DC_\alpha(\tilde{\gamma}, \tilde{\tau})$ .

Since  $\nu > \tau$ , we find that  $N'_i/N_{n_i}$  goes to infinity, and therefore, for  $i$  big enough the spectral support of  $H_i(\cdot) = D_i(\cdot)D^*(\cdot)$  (i.e. of the diverging sequence of conjugations) is contained in  $[-N'_i, N'_i]^d \subset \mathbb{Z}^d$ . In a similar way, and with some obvious notation, we find that

$$(e^{2i\pi(k\alpha+a_d)} - 1)\hat{D}_i^{h,j}(k) = O(\varepsilon_{n,0})$$

Consequently,

$$\dot{T}_{N'_i} D_i(\cdot) = O_{L^2}(\varepsilon_{n,0})$$

On the other hand, since  $\sigma(H_i(\cdot)) \subset [-2N_{n_i}, 2N_{n_i}]^d$  and since  $N_{n_i} \ll N'_i$ .

$$\begin{aligned} T_{N'_i/2} D(\cdot) &= T_{N'_i/2} [D_i(\cdot) D_i^*(\cdot) D(\cdot)] \\ &= T_{N'_i/2} [T_{N'_i} (D_i(\cdot)) H_i^*(\cdot)] \\ &= C_i H_n^*(\cdot) + O_{L^2}(\varepsilon_{n,0}) \end{aligned}$$

Since for  $n$  big enough  $\|D(\cdot) - T_{N'_i} D(\cdot)\|_{L^2}$  is small and  $D_i^*(\cdot) D(\cdot)$  takes values in  $Inn(su(2)) \approx SO(3)$ , we can assume that  $C_i \in L(su(2))$  is bounded away from 0, say

$$|C_i| > \frac{1}{2}$$

in operator norm. Since  $H_i^*(\cdot)$  diverges in  $L^2$ , we reach a contradiction.

### The case of a general compact Lie group

The case of a general compact group is hardly more complicated. If the sequence of conjugations diverges, there exists a root  $\rho$  such that  $|a_\rho^{(i)}|_{\mathbb{Z}} \leq K_{n_i}^{-1}$ , where  $a_\rho = \rho(a)$ . If we fix such a root  $\rho$ , we find that for any other positive root  $\rho'$ ,

$$\begin{aligned} (e^{2i\pi(k\alpha+a_\rho^{(n)}-a_\rho^{(d)})} - 1) e^{2i\pi a_\rho^{(d)}} \hat{D}_i^{j_\rho, j_{\rho'}}(k) &= O(\varepsilon_{n,0}) \\ (e^{2i\pi(k\alpha+a_d)} - 1) \hat{D}_i^{h_{\rho'}, j_\rho}(k) &= O(\varepsilon_{n,0}) \end{aligned}$$

and, in the same way as before,

$$\dot{T}_{N'_i} D_i(\cdot).j_\rho = O_{L^2}(\varepsilon_{n_i,0})$$

and since

$$\begin{aligned} T_{N'_i} D(\cdot).j_\rho &= T_{N'_i} [D_i(\cdot) D_i^*(\cdot) D(\cdot).j_\rho] \\ &= T_{N'_i} [T_{2N'_i}(D_i(\cdot)) D_i^*(\cdot) B(\cdot).j_\rho] \\ &= C_i D_i^*(\cdot) D(\cdot).j_\rho + O_{L^2}(\varepsilon_{n_i,0}) \end{aligned}$$

Since  $D(\cdot)$  is an isometry, we find that  $C_i \in L(g)$  is bounded away from 0, say

$$|C_i| > \frac{1}{2}$$

in operator norm. Now,  $D_i^*(\cdot) D(\cdot).j_\rho$  diverges in  $L^2$  as  $j_\rho$  does not commute with the reduction of resonant modes. This is due to the fact that, by construction of  $h_i$  and by the choice of the root  $\rho$

$$[h_i, j_\rho] = 2i\pi \frac{k'_\rho}{D}$$

infinitely often, with  $k'_\rho \rightarrow \infty$ . Thus, the hypothesis that the product of conjugation diverges leads us to a contradiction.

Finally, we observe that if  $U(\cdot)$  is small enough so that the K.A.M. scheme can be applied, the diophantine condition on  $A_d$  becomes irrelevant. If we suppose that  $A_d \in DC_\alpha(\tilde{\gamma}, \tau')$  with  $\tau' > \tau$ , then, after a finite number of iterations of the scheme,  $F_i(\cdot)$  is small enough so that the scheme can be initiated if we place  $\alpha$  in  $DC(\gamma, \tau')$ , and the argument presented above remains valid, and this concludes the proof of the theorem in its full generality.

### 5.3 Reducibility to a Liouvillean constant

A corollary of this proof is, in fact, the optimality of the scheme in the orbits of Diophantine constant cocycles. By its construction, the scheme converges in the smooth category if, and only if, it converges in  $L^2$ , and the proof implies that if a measurable conjugation to such a constant exists, then the scheme converges toward it, eventually modulo a conjugation between constant cocycles.

On the other hand, the transposed argument shows that the K.A.M. scheme is highly non-optimal if the dynamics in the fibres are Liouvillean. More precisely, we let  $(\alpha, Ae^{F(\cdot)}) \in SW^\infty(\mathbb{T}, SU(2))$  be smoothly conjugate to  $(\alpha, A_L)$ , where  $A_L$  is a Liouvillean constant in  $SU(2)$ . Application of the scheme produces a sequence of conjugations  $F_n(\cdot)$  and a sequence of cocycles  $(\alpha, A_n e^{F_n(\cdot)})$  such that

$$Ae^{F(\cdot)} = H_n(\cdot + \alpha) A_n e^{F_n(\cdot)} H_n^*(\cdot)$$

where  $F_n(\cdot) \rightarrow 0$  exponentially fast. If we suppose that the sequence of conjugations converges, we find that in the limit

$$A_L = \tilde{H}(\cdot + \alpha)A_\infty \tilde{H}^*(\cdot)$$

where  $A_\infty$  (which we suppose diagonal, just as  $A_L$ ) is the limit of  $A_n$ . Since  $A_L$  is non-resonant,  $\tilde{H}(\cdot)$  is a torus morphism, so that  $A_\infty$  is Liouvillean itself. Since, now,  $F_n(\cdot) \rightarrow 0$  exponentially fast and, for  $n$  big enough,  $A_{n+1} = A_n \exp(\hat{F}_n(0))$ , we can rewrite  $A_n e^{F_n(\cdot)}$  as  $A_\infty e^{\tilde{F}_n(\cdot)}$  where still  $\tilde{F}_n(\cdot) \rightarrow 0$  exponentially fast. Since  $A_\infty$  is Liouvillean, for any  $l \in \mathbb{N}$ , there exists  $k_l$  such that

$$|a_\infty - k_l \alpha| < \frac{1}{|k_l|^l}$$

Therefore the Fourier mode  $k_l$  is a resonance for the scheme at the  $n$ -th step provided that

$$\begin{aligned} \frac{1}{|k_l|^l} &< N_n^{-\nu} \\ |k_l| &< N_n \end{aligned}$$

or equivalently

$$N_n^{\nu/l} < |k_l| < N_n$$

Therefore, since for  $l > \nu$  a reduction of resonant mode must take place, which contradicts the hypothesis that  $H_n(\cdot)$  converges. Since  $l$  can be chosen arbitrarily big, no choice of  $\nu$  can make the scheme converge.

Strictly speaking, this phenomenon appears for "generic" reducible cocycles. Genericity here is in the product topology  $G \times C^\infty(\mathbb{T}^d, G)$ , which is very far from the one induced by  $SW^\infty$ . Genericity in this sense comes from the demand that resonant modes be non-0 infinitely often, in order to avoid trivialities. A converging K.A.M. scheme can nonetheless be defined, but we refer the reader to [Kar14] for its construction.

## References

- [AFK01] A. ARTHUR, B. FAYAD, and R. KRIKORIAN, A KAM scheme for  $SL(2, R)$  cocycles with Liouvillean frequencies, *arXiv* (2001).
- [AK06] A. AVILA and R. KRIKORIAN, Reducibility or nonuniform hyperbolicity for quasiperiodic Schrödinger cocycles, *Ann. of Math. (2)* **164** (2006), 911–940. MR 2259248 (2008h:81044).
- [Eli02] L. H. ELIASSEN, Ergodic skew-systems on  $T^d \times SO(3, R)$ , *Ergodic Theory and Dynamical Systems* **22** (2002), 1429–1449.
- [FK09] B. FAYAD and R. KRIKORIAN, Rigidity results for quasiperiodic  $SL(2, \mathbb{R})$ -cocycles, *J. Mod. Dyn.* **3** (2009), 497–510. MR 2587083 (2011f:37046).

- [HY09] X. HOU and J. YOU, Local rigidity of reducibility of analytic quasi-periodic cocycles on  $U(n)$ , *Discrete Contin. Dyn. Syst.* **24** (2009), 441–454. MR 2486584 (2010e:37024). <http://dx.doi.org/10.3934/dcds.2009.24.441>.
- [Kar13] N. KARALIOLIOS, *Aspects globaux de la réductibilité des cocycles quasi-périodiques à valeurs dans des groupes de Lie compacts semi-simples*, 2013. Available at <http://tel.archives-ouvertes.fr/docs/00/77/79/11/PDF/these.pdf>.
- [Kar14] N. KARALIOLIOS, *Invariant distributions for quasiperiodic cocycles in  $T^d \times SU(2)$ , in preparation*, 2014. arXiv 1407.4763.
- [Kol49] A. KOLMOGOROFF, On inequalities between the upper bounds of the successive derivatives of an arbitrary function on an infinite interval, *Amer. Math. Soc. Translation* **1949** (1949), 19. MR 0031009 (11,86d).
- [Kri99] R. KRIKORIAN, *Réductibilité des systèmes produits-croisés à valeurs dans des groupes compacts*, no. 259, 1999. MR 1732061 (2001f:37030).
- [Kri01] R. KRIKORIAN, Global density of reducible quasi-periodic cocycles on  $T^1 \times SU(2)$ , *Annals of Mathematics* **154** (2001), 269–326.